Lecture 4
Continuous-time Queues

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Outline

- Continuous-time Markov Chains
  - Reading: Section 9.1 of Srikant & Ying

- Little's Law & M/M/1 Queue

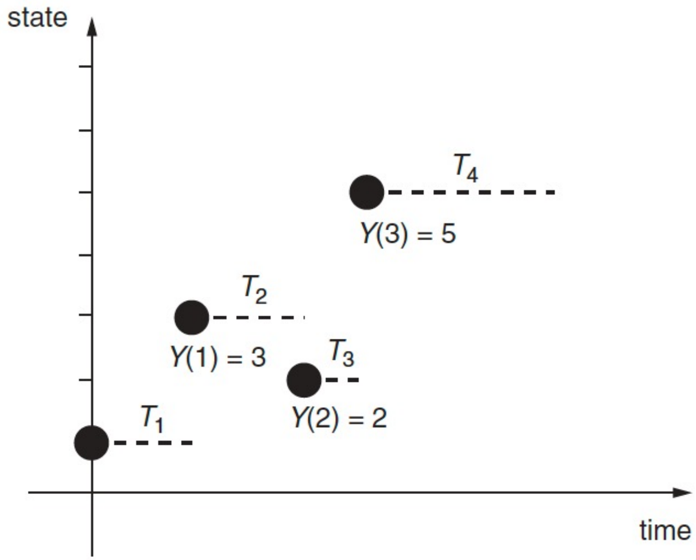
  - R. Srikant and Lei Ying, *Communication Networks: An Optimization Control and Stochastic Networks Perspective*, Cambridge University Press, 2014.

Continuous-time Markov Chains

- A stochastic process $\{X(t)\}$ is a *Continuous-Time Markov Chain (CTMC)* if the following conditions hold:
  - (i) t belongs to some interval of $\mathcal{R}$;
  - (ii) $X(t) \in S$, where S is a countable set; and
  - (iii) $\Pr(X(t+s)|X(u), u \leq s) = \Pr(X(t+s)|X(s))$, i.e., the conditional probability distribution at time $t+s$, given the complete history up to time s, depends only on the state of the process at time s.

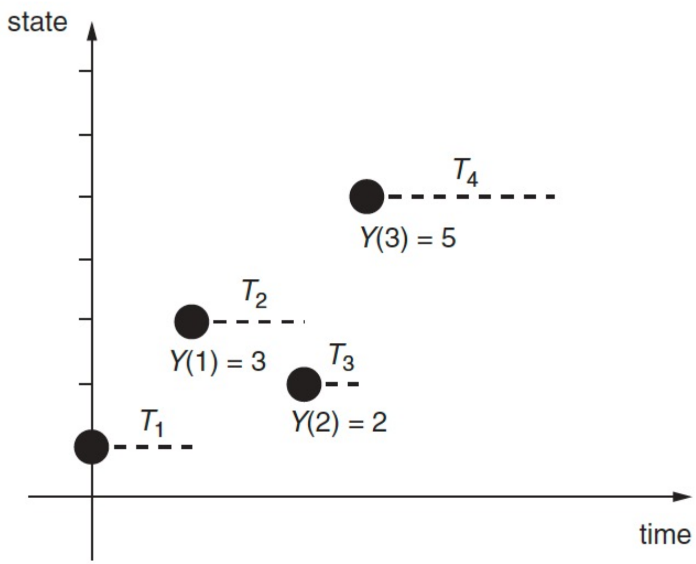
- A CTMC is said to be *time homogeneous* if $\Pr(X(t+s)|X(s))$ is independent of s, i.e., the conditional probability distribution is time independent.

Time to Stay in a State



- **Theorem 9.1.1** The time to stay in any state $i \in S$ of a time-homogeneous CTMC is exponentially distributed.
  - Proof. See p. 230 of Srikant & Ying

Time to Stay in a State (2)



- Time-homogeneous CTMC can also be described as follows.
  - (i) During each visit to state i, the CTMC spends an exponentially distributed amount of time in this state. We let $1/q_i$ denote the mean time spent in state i.
  - (ii) After spending an exponentially distributed amount of time in state i, the CTMC moves to state $j \neq i$ with probability P_{ij}.

Non-explosiveness CTMCs

- We assume that the number of transitions in a finite time interval is finite with probability 1.
- One can construct CTMCs with strange behavior if we do not assume this condition.
- This condition is called *non-explosiveness*.

Transition Rate Matrix

- Let $p(t)$ denote a vector of probabilities with

$$p_i(t) = \Pr(X(t) = i).$$

- There exists a transition rate matrix $\boldsymbol{Q}$ such that

$$\dot{p}_i(t) = Q_{ii}\, p_i(t) + \sum_{j \neq i} Q_{ji}\, p_j(t)$$

  - Written in matrix form:

$$\dot{p}(t) = p(t)\boldsymbol{Q}$$

  - See p. 231 Srikant & Ying for derivations
- Transition rate matrix $\boldsymbol{Q}$ completely describes the CTMC, along with $p(0)$.
  - Given $\boldsymbol{Q}$, we need not specify q_i and P_{ij}

Transition Rate Matrix (2)

- Transition rate matrix $\boldsymbol{Q}$ satisfies

$$Q_{ii} \leq 0$$

$$Q_{ij} \geq 0 \text{ for } i \neq j$$

$$\sum_j Q_{ij} = 0$$

- **Intuition:**
  - Probability reduction in state i, due to transitions from state i to the other states = total probability growth of the other states, due to transitions from state i to the other states

Key Questions

The following questions are important in the study of CTMC.

- Does there exist a distribution vector π so that $0 = \pi \boldsymbol{Q}$?
  - If it exists, it is called a stationary distribution.

- If there exists a unique stationary distribution, does convergence $\lim_{t \to \infty} p(t) = \pi$ hold for all $p(0)$?

Irreducible CTMCs

Definition 9.1.1 (Irreducibility) A CTMC is said to be irreducible if, given any two states i and j,

$$\Pr(X(t) = j | X(0) = i) > 0$$

for some finite t. □

- Note that there is no concept of aperiodicity for CTMCs because state transitions can happen at any time.

Finite-State-Space CTMCs

- The following theorem states that a CTMC has a unique stationary distribution if it is irreducible and has a finite state space.

Theorem 9.1.2 A finite-state-space, irreducible CTMC has a unique stationary distribution π and $\lim_{t\to\infty} p(t) = \pi, \forall p(0)$. □

- For a finite-state-space, irreducible CTMC, the stationary distribution can be computed by finding a $\pi \geq 0$ such that $\pi \boldsymbol{Q} = 0$ and $\sum_i \pi_i = 1$.

- Finite state space + irreducible + CTMC➔ existence + uniqueness + convergence to stationary distribution

Infinite-State-Space CTMCs

- If the state space is infinite, irreducibility is not sufficient to guarantee that the CTMC has a unique stationary distribution.

- Exercise 1. Consider a CTMC $X(t)$ with the state space to be integers. The transition rate matrix $\boldsymbol{Q}$ is

$$Q_{ij} = \begin{cases} 1, & \text{if } j = i + 1, \\ 1, & \text{if } j = i - 1, \\ -2, & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

  - Is it irreducible?
  - What is its stationary distribution?

Positive Recurrent CTMCs

- Similar to DTMCs, we introduce the notion of recurrence and conditions beyond irreducibility to ensure the existence of stationary distributions.

Definition 9.1.2 Assuming $X(0) = i$, the *recurrence time* is the first time the CTMC returns to state i after it leaves the state. Recall that the amount of time spent in state i is defined as

$$\gamma_i = \inf\{t > 0 : X(t) \neq i \text{ and } X(0) = i\}.$$

The recurrence time τ_i is defined as

$$\tau_i = \inf\{t > \gamma_i : X(t) = i \text{ and } X(0) = i\}.$$

State i is called *recurrent* if

$$\Pr(\tau_i < \infty) = 1,$$

and transient otherwise.

A recurrent state is *positive recurrent* if $E[\tau_i] < \infty$, and is *null recurrent* if $E[\tau_i] = \infty$. □

Positive Recurrent CTMCs (2)

Lemma 9.1.3 For an irreducible CTMC, if one state is positive recurrent (null recurrent), all states are positive recurrent (null recurrent). Further,

$$\lim_{t\to\infty} p_i(t) = \frac{1}{E[\tau_i](-Q_{ii})},$$

which holds even when $E[\tau_i] = \infty$. □

Theorem 9.1.4 Consider an irreducible and non-explosive CTMC. A unique stationary distribution $\pi \geq 0$ (i.e., $\sum_i \pi_i = 1$ and $\pi\mathbf{Q} = 0$) exists if and only if the CTMC is positive recurrent. □

This theorem states that if we can find a vector $\pi \geq 0$ such that $\pi\mathbf{Q} = 0$ and $\sum_i \pi_i < \infty$, and the CTMC is non-explosive, the CTMC is positive recurrent. Note that if $\sum_i \pi_i \neq 1$, we can define $\tilde{\pi}_i = \pi_i / \sum_j \pi_j$, and $\tilde{\pi}$ is a stationary distribution.

Theorem 9.1.5 If there exists a π such that $\pi\mathbf{Q} = 0$ and $\sum_i \pi_i = \infty$ for an irreducible CTMC, the CTMC is not positive recurrent and

$$\lim_{t\to\infty} p_i(t) = 0$$

for all i. □

Global Balance Equation

- The following lemma presents an alternative characterization of the equation that has to be satisfied by the stationary distribution of a CTMC.

Lemma 9.1.6 $\pi \mathbf{Q} = 0$ is equivalent to

$$\sum_{i \neq j} \pi_i Q_{ij} = \pi_j \sum_{i \neq j} Q_{ji}, \quad \forall j. \tag{9.1}$$

- This equation is called the global balance equation.
- **Intuition:**
  - total rate of transitions into state *j* = total rate of transitions out of state *j*.
  - Proven based on $\sum_j Q_{ij} = 0$

Local Balance Equation

- Local Balance Equation is a sufficient condition of the global balance equation

Lemma 9.1.7 The global balance equation holds if

$$\pi_i Q_{ij} = \pi_j Q_{ji}, \qquad \forall i \neq j. \tag{9.2}$$

- **Intuition:**
  - Rate of transitions from state i to state j = rate of transitions from state j to state i
  - It is possible to have a stationary distribution π that satisfies the global balance equation, but not the local balance equation.

Foster-Lyapunov Theorem

- Often it is difficult to find the π to satisfy either the global or the local balance equation.
- In the applications it is important to know whether π exists, even if we cannot find it explicitly.
- Similar to the Foster–Lyapunov theorem for DTMCs, the following Foster–Lyapunov theorem for CTMCs provides another sufficient condition for a CTMC to be positive recurrent.

Theorem 9.1.8 (Foster–Lyapunov theorem for CTMCs) Suppose $X(t)$ is irreducible and non-explosive. If there exists a function $V : \mathcal{S} \to \mathcal{R}^+$ such that

(1) $\sum_{j \neq i} Q_{ij}(V(j) - V(i)) \leq -\epsilon$ if $i \in \mathcal{B}^c$, and
(2) $\sum_{j \neq i} Q_{ij}(V(j) - V(i)) \leq M$ if $i \in \mathcal{B}$,

for some $\epsilon > 0$, $M < \infty$, and a bounded set $\mathcal{B}$, then $X(t)$ is positive recurrent.

Summary

- Transition rate matrix $\boldsymbol{Q}$, $\dot{p}(t) = p(t)\boldsymbol{Q}$
- Key questions for CTMCs
  - Existence of stationary distribution ?
  - Convergence to unique stationary distribution ?
- Finite-state-space CTMCs
  - Irreducible ➔ existence + uniqueness + convergence
- Countable-state-space CTMCs
  - Foster-Lyapunov ➔ positive recurrent
  - Irreducible + positive recurrent ➔ existence + uniqueness + convergence
- Reading: Section 9.1 of Srikant & Ying